Math 522 Exam 10 Solutions

1. Factor 50!. Hint: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47

   BONUS: Factor $\binom{50}{25}$.

   > $50! = 2^{a_2}3^{a_3}5^{a_5}7^{a_7}\cdots 47^{a_{47}}$. We now apply Thm. 8-6 repeatedly to find the exponents. $a_2 = \lfloor\frac{50}{2}\rfloor+\lfloor\frac{50}{4}\rfloor+\lfloor\frac{50}{8}\rfloor+\lfloor\frac{50}{16}\rfloor+\lfloor\frac{50}{32}\rfloor = 47, a_3 = \lfloor\frac{50}{3}\rfloor+\lfloor\frac{50}{9}\rfloor+\lfloor\frac{50}{27}\rfloor = 22, a_5 = \lfloor\frac{50}{5}\rfloor + \lfloor\frac{50}{25}\rfloor = 12, a_7 = \lfloor\frac{50}{7}\rfloor + \lfloor\frac{50}{49}\rfloor = 8, a_{11} = \lfloor\frac{50}{11}\rfloor = 4, a_{13} = \lfloor\frac{50}{13}\rfloor = 3, a_{17} = \lfloor\frac{50}{17}\rfloor = 2, a_{19} = \lfloor\frac{50}{19}\rfloor = 2, a_{23} = \lfloor\frac{50}{23}\rfloor = 2, a_{29} = \cdots = a_{47} = 1.$
   >
   > Hence $50! = 2^{47}3^{22}5^{12}7^{8}11^{4}13^{3}17^{2}19^{2}23^{2}29^{1}31^{1}37^{1}41^{1}43^{1}47^{1}$.
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   > BONUS: We set $25! = 2^{b_2}3^{b_3}5^{b_5}7^{b_7}\cdots 47^{b_{47}}$, and find $b_2 = \lfloor\frac{25}{2}\rfloor+\lfloor\frac{25}{4}\rfloor+\lfloor\frac{25}{8}\rfloor+\lfloor\frac{25}{16}\rfloor = 22, b_3 = \lfloor\frac{25}{3}\rfloor + \lfloor\frac{25}{9}\rfloor = 10, b_5 = \lfloor\frac{25}{5}\rfloor + \lfloor\frac{25}{25}\rfloor = 6, b_7 = \lfloor\frac{25}{7}\rfloor = 3, b_{11} = \lfloor\frac{25}{11}\rfloor = 2, b_{13} = \cdots = b_{23} = 1.$
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   > Hence $\binom{50}{25} = 2^{47-44}3^{22-20}5^{12-12}7^{8-6}11^{4-4}13^{3-2}17^{2-2}19^{2-2}23^{2-2}29^{1}31^{1}37^{1}41^{1}43^{1}47^{1} =$
   >
   > $= 2^{3}3^{2}7^{2}13^{1}29^{1}31^{1}37^{1}41^{1}43^{1}47^{1}$

2. Prove that for all $x > (121)^{120}$, there is a prime between x and $121x$.

   > We first prove the lemma that for $x > (121)^{120}$, $\frac{30.25}{\ln(121x)} > \frac{30}{\ln(x)}$. Indeed, taking logs, $\ln x > 120 \ln 121$, so $0.25 \ln x > 30 \ln 121$ and $30.25 \ln x > 30 \ln x + 30 \ln 121 = 30 \ln(121x)$, from which the lemma follows. So, using Thm. 8-8 twice, $\pi(121x) - \pi(x) > \frac{\ln 2}{4}\frac{121x}{\ln 121x} - 30 \ln 2 \frac{x}{\ln x} = x \ln 2 \left(\frac{30.25}{\ln 121x} - \frac{30}{\ln x}\right)$. For $x > (121)^{120}$, we apply the lemma, and hence this is greater than zero, so $\pi(121x) > \pi(x)$ so there must be a prime between x and $121x$.

3. High score=99, Median score=65, Low score=52